Analysis Notes (2023/2024)

Griffin Reimerink

Contents

1 Bounds

$A \subseteq R$ is **bounded above** if $\exists b \in \mathbb{R}$ such that $a \leq b \quad \forall a \in A$ where b is an **upper bound**.
$A \subseteq R$ is **bounded below** if $\exists l \in \mathbb{R}$ such that $a \geq l \quad \forall a \in A$ where l is a **lower bound**.
$s \in \mathbb{R}$ is the **supremum** of A ($\sup A$) if s is an upper bound and $s \leq b$ for any upper bound b of A.
$i \in \mathbb{R}$ is the **infimum** of A ($\inf A$) if i is an upper bound and $i \geq l$ for any lower bound l of A.
The supremum is also the **least upper bound**, the infimum is also the **greatest lower bound**.

A supremum is not always a maximum and an infimum is not always a minimum.

If s is an upper bound for A, then $\quad s = \sup A \iff \forall \epsilon > 0 \;\; \exists a \in A$ such that $s - \epsilon < a$
If i is a lower bound for A, then $\quad i = \inf A \iff \forall \epsilon > 0 \;\; \exists a \in A$ such that $a < i + \epsilon$

1.1 Completeness

Axiom of Completeness:
every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound

Archimedean Property:

1. $\forall x \in \mathbb{R} \;\; \exists n \in \mathbb{N}$ s.t. $n > x$
2. $\forall y > 0 \;\; \exists n \in \mathbb{N}$ s.t. $1/n < y$

Nested Interval Property: $[a_1, b_1] \subseteq [a_2, b_2] \subseteq [a_3, b_3] \subseteq \ldots \implies \bigcap_{n=1}^{\infty} [a_n, b_n] \neq \varnothing$
This is only true for closed intervals.

The rational numbers are dense in $\mathbb{R}$: there is a rational number between every two real numbers.

1.2 Absolute values

$$\begin{cases} |x| = x \text{ if } x \geq 0 \\ |x| = -x \text{ if } x < 0 \end{cases} \qquad |x| = \max -x, x \qquad |ab| = |a||b| \qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

If $x \in \mathbb{R}$ and $x > 0$: $\quad |x| \leq a \iff -a < x < a \quad$ and $\quad |x| \geq a \iff x \geq a \vee x \leq -a$

Triangle inequality: $|a + b| \leq |a| + |b| \qquad$ **Reverse triangle inequality**: $||a| - |b|| \leq |a - b|$

2 Functions

A **function** $f : A \to B$ is a rule that assigns to each $a \in A$ a single element $b = f(a) \in B$.
A is the **domain**, B is the **co-domain**, $f(A) = \{f(a) : a \in A\}$ is the **range** of f.

A function $f : A \to B$ is called

1. **injective** if $f(a) = f(b) \implies a = b$
2. **surjective** if $B = f(A)$, i.e. $\quad \forall b \in B \;\; \exists a \in A$ s.t. $b = f(a)$
3. **bijective** if f is both injective and surjective

Two sets have the same cardinality of there exists a bijective function $f : A \to B$ (notation: $A \sim B$)
1 to 1 correspondence is an equivalence relation.

The **power set** $\mathcal{P}(A)$ is the set of all subsets of A.
Cantor's theorem: There is no surjective function $f : A \to \mathcal{P}(A)$.
As a consequence, $\#\mathbb{N} < \#\mathcal{P}(\mathbb{N}) < \#\mathcal{P}(\mathcal{P}(\mathbb{N}))\ldots$ therefore there exist infinitely many infinities.

2.1 Countability

A set A is **countable** if $A \sim S$ for some $S \subseteq \mathbb{N}$ and **uncountable** otherwise.

The following statements are equivalent: ($\mathbb{N}$ can be replaced by any countable set)

1. A is countable
2. there exists an injective map $g : A \to \mathbb{N}$
3. there exists a surjective map $h : \mathbb{N} \to A$

The union, Cartesian product and subsets of countable sets are countable.
A_n is countable for all $n \in \mathbb{N} \implies \bigcup_{n=1}^{\infty} A_n$ is countable
$\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable. $\mathbb{R}$, $(0,1)$ and $\mathbb{Q}^c$ are uncountable.

3 Sequences

A **sequence** is a function with domain $\mathbb{N}$. They can be defined recursively.

a_n **converges** to a if a_n gets arbitrarily close to a as n grows larger.
Notation: a $= \lim_{n\to\infty} a_n$ or $a = \lim a_n$ or $a_n \to a$

ϵ-neighborhood of a: For $a \in \mathbb{R}$ and $\epsilon > 0$, $V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$.
(a_n) converges to a if:

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad n \geq N \implies |a_n - a| < \epsilon \quad (\text{or } a_n \in V_\epsilon(a))$$

To use this definition you need an educated guess for a.

(a_n) is **divergent** if it does not converge:

$$\exists \epsilon > 0 \quad \text{s.t.} \quad \forall N \in \mathbb{N} \quad \exists n \geq N \quad \text{s.t.} \quad |a_n - a| \geq \epsilon$$

(a_n) is **bounded** if:

$$\exists M > 0 \quad \text{s.t.} \quad |a_n| \leq M \quad \forall n \in \mathbb{N}$$

(a_n) is convergent $\implies$ (a_n) is bounded (but not the other way around)

(a_n) is **monotone** if it is either increasing ($a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$) or decreasing ($a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$).
(a_n) is bounded and monotone $\implies$ (a_n) converges.
If a_n is decreasing, $\lim a_n = \sup A$ If a_n is increasing, $\lim a_n = \inf A$

(a_n) is a **Cauchy sequence** if

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad n, m \geq N \implies |a_n - a_m| < \epsilon$$

(a_n) is convergent $\iff$ (a_n) is a Cauchy sequence $\implies$ (a_n) is bounded ($\impliedby$ only holds for $\mathbb{R}$)

3.1 Limits

$$\lim \frac{1}{n^a} = 0 \quad (a > 0) \qquad \lim c^n = 0 \quad (-1 < c < 1) \qquad \lim c^n n^a = 0 \quad (-1 < c < 1,\, a \in \mathbb{R})$$

$$\lim \sqrt[n]{c} = 1 \quad (c > 0) \qquad \lim \sqrt[n]{n} = 1 \qquad \lim \frac{n!}{n^n} = 0$$

Algebraic Limit Theorem
If $a = \lim a_n$ and $b = \lim b_n$, then
$\lim(ca_n) = ca \quad (c \in \mathbb{R})$ $\qquad \lim(a_n + b_n) = a + b$ $\qquad \lim(a_n b_n) = ab$ $\qquad \lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b} \quad (b \neq 0)$

Order Limit Theorem
For all $n \in \mathbb{N}$:

$$a_n \geq 0 \implies a \geq 0 \qquad a_n \leq b_n \implies a \leq b$$

For all $n \in \mathbb{N}$ and $c \in \mathbb{R}$

$$c \leq b_n \implies c \leq b \qquad a_n \leq c \implies a \leq c$$

3.2 Subsequences

Definition: Pick $n_k \in \mathbb{N}$ such that $1 \leq n_1 < n_2 < \dots$ (Note: $n_k \geq k$ for all $k \in \mathbb{N}$)
If (a_n) is a sequence then $(a_{n_k}) = (a_{n_1}, a_{n_2}, \dots)$ is a **subsequence** of n.

$\lim a_n = a \implies \lim a_{n_k} = a$
Two different subsequences of a_n have different limits $\implies$ a_n diverges.

Bolzano-Weierstrass theorem: Every bounded sequence has a convergent subsequence.

3.3 Series

Infinite **series**: $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots$ n-th **partial sum**: $s_n = a_1 + a_2 + \dots + a_n$
If $\lim s_n = s$, then we say the series converges to s.
$\forall n > 1 \quad \sum_{k=1}^{\infty} \frac{1}{k^n}$ converges, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Assume that $f : [1, \infty) \to \mathbb{R}$ is positive, continuous and monotonically decreasing.
Let $a_k = f(x)$, then $\sum_{k=1}^{\infty} a_k$ converges $\iff \int_1^{\infty} f(x)\, dx < \infty$

Algebraic limit theorem for series:

$$\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k \quad \forall c \in \mathbb{R} \qquad \sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

Cauchy criterion for series: $\sum_{k=1}^{\infty} a_k$ converges if and only if

$$\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \text{s.t.} \;\; n > m \geq N \implies |a_{m+1} + a_{m+2} + \dots + a_n| < \epsilon$$

$$\sum_{k=1}^{\infty} a_k \text{ converges} \implies \lim a_k = 0 \qquad \lim a_k \neq 0 \implies \sum_{k=1}^{\infty} a_k \text{ diverges}$$

If $0 \leq a_k \leq b_k$ for all (sufficiently large) $k \in \mathbb{N}$, then

$$\sum_{k=1}^{\infty} b_k \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges} \qquad \sum_{k=1}^{\infty} a_k \text{ diverges} \implies \sum_{k=1}^{\infty} b_k \text{ diverges}$$

Assume $0 \leq a_{k+1} \leq a_k \;\; \forall k \in \mathbb{N}$ and $\lim a_k = 0$.
Then the **alternating series** $\sum_{k=1}^{\infty}(-1)^{k+1} a_k$ converges.

$$\sum_{k=1}^{\infty} |a_k| \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges}$$

A convergent series is **absolutely convergent** if it still converges when a_k is replaced by $|a_k|$. Otherwise it is **conditionally convergent**.

A **geometric series** is of the form $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$

$$s_n = a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \qquad \text{For } |r| < 1: \quad \lim s_n = \lim \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$$

Telescoping series are of the form $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(b_k - b_{k+1})$ $s_n = a_1 + a_2 + \dots + a_n = b_1 - b_{n+1}$
If a telescoping series a_n converges, b_n also converges.

4 Properties of sets

Closed interval: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ **Open interval**: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

4.1 Open sets

$O \subseteq \mathbb{R}$ is an **open set** if you can fit an ϵ-neighborhood around every element:

$$\forall a \in O \quad \exists \epsilon > 0 \quad \text{s.t.} \quad V_\epsilon(a) \subseteq O$$

The empty set is open by definition. An open interval is also an open set.

Unions of (finite or infinite) collections of open sets are open.
Intersections of finite collections of open sets are open.

4.2 Limit points

$a \in A$ is an **isolated point** of $A \subseteq \mathbb{R}$ if

$$\exists \epsilon > 0 \quad \text{s.t.} \quad V_\epsilon(a) \cap A = \{a\}$$

$x \in \mathbb{R}$ is a **limit point** of $A \subseteq \mathbb{R}$ if

$$\forall \epsilon > 0 \quad \exists a \neq x \quad \text{s.t.} \quad \{a\} \in V_\epsilon(x) \cap A$$

The following statements are equivalent:

1. x is a limit point of A
2. There exists a sequence (a_n) in A such that $a_n \neq x \quad \forall n \in \mathbb{N}$ and $x = \lim a_n$

4.3 Closed sets

A set is **closed** if it contains all of its limit points. A closed interval is a closed set.

The following statements are equivalent:

1. F is closed
2. Every Cauchy (or convergent) sequence in F has its limit in F

The **closure** of A is defined as $\overline{A} = A \cup \{\text{all limit points of } A\}$. $\overline{A}$ is closed.

Unions of finite collections of closed sets are closed.
Intersections of (finite or infinite) collections of closed sets are closed.

$$O \text{ open} \iff O^c \text{ closed} \qquad F \text{ closed} \iff F^c \text{ open}$$

4.4 Compact sets

$K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a convergent subsequence
whose limit is contained in K.

Examples of compact sets: every finite set, closed intervals
Examples of non-compact sets: non-closed intervals, $\mathbb{R}$

$$K \subseteq \mathbb{R} \text{ is compact} \iff K \text{ is closed and bounded}$$

Assume that $K_n \neq \varnothing$ is compact for all $n \in \mathbb{N}$ and $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

4.5 Open covers

Let $A \subseteq \mathbb{R}$ and assume that the sets $O_\lambda \subseteq \mathbb{R}$, where $\lambda \in \Lambda$, are open.
We call the sets O_λ an **open cover** if $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$

Heine-Borel Theorem
Let $K \in \mathbb{R}$. The following statements are equivalent:

1. K is compact
2. K is closed and bounded
3. Any open cover for K has a finite subcover

5 Limits of functions

Let $f : A \to \mathbb{R}$ and c a limit point of A. We say that $\lim_{x \to c} f(x) = L$ when

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \quad \text{s.t.} \quad (x \in A \quad \text{and} \quad 0 < |x - c| < \delta) \implies |f(x) - L| < \epsilon$$

where δ may depend on ϵ and c.

The following statements are equivalent:

1. $\lim_{x \to c} f(x) = L$
2. $\lim f(x_n) = L$ for all $(x_n) \subseteq A$ with $x_n \neq c$ and $\lim x_n = c$

Let $f, g : A \to \mathbb{R}$, c a limit point of A, and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then:

1. $\lim_{x \to c} kf(x) = kL$ $(k \in \mathbb{R})$
2. $\lim_{x \to c} f(x) + g(x) = L + M$
3. $\lim_{x \to c} f(x)g(x) = LM$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ where $g(x) \neq 0$

5.1 Continuity

$f : a \to \mathbb{R}$ is **continuous** at $c \in A$ if

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \quad \text{s.t.} \quad (x \in A \quad \text{and} \quad 0 < |x - c| < \delta) \implies |f(x) - f(c)| < \epsilon$$

where $f(c)$ needs to be defined, but c does not need to be a limit point of A.

Let $f : A \to \mathbb{R}$ and $c \in A$. The following statements are equivalent:

1. f is continuous at c
2. $\lim f(x_n) = f(x)$ for all $(x_n) \subseteq A$ with $x_n \neq c$ and $\lim x_n = c$
3. If c is a limit point of A: $\lim_{x \to c} f(x) = f(c)$

$f : A \to \mathbb{R}$ continuous and $K \subseteq A$ compact $\implies$ $f(K)$ compact (not true for inverses)

Let $K \subseteq \mathbb{R}$ be compact and $f : K \to \mathbb{R}$ continuous, then f attains a maximum and minimum on K.

$f : A \to \mathbb{R}$ is **uniformly continuous** (δ does not depend on x or y) if

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \quad \text{s.t.} \quad \forall x, y \in A \qquad |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Logical negation: $f : A \to \mathbb{R}$ is not uniformly continuous if

$$\exists \epsilon_0 > 0 \quad \text{s.t.} \quad \forall \delta > 0 \;\; \exists x, y \in A \;\; \text{for which} \;\; |x - y| < \delta \;\; \text{but} \;\; |f(x) - f(y)| \geq \epsilon_0$$

The following statements are equivalent:

1. $f : A \to \mathbb{R}$ is not uniformily continuous on A
2. $\exists \epsilon_0 > 0$ and $(x_n), (y_n) \subseteq A \quad$ s.t. $\quad |x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all n.

If $f : K \to \mathbb{R}$ is continuous and K is compact then f is uniformily continuous on K.

Intermediate Value Theorem:
If $f : [a, b] \to \mathbb{R}$ is continuous and L is between $f(a)$ and $f(b)$, then $f(c) = L$ for some $c \in (a, b)$

5.2 Derivatives

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. Then f is **differentiable** at $c \in I$ if

$$f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists}$$

$f : I \to \mathbb{R}$ differentiable at $c \in I \implies f$ continuous at c

Interior extremum theorem:
Assume $f : (a, b) \to \mathbb{R}$ is differentiable and f attains a maximum or minimum at $c \in (a, b)$.
Then, $f'(c) = 0$. (need not be true for closed intervals)

Darboux's Theorem:
Assume $f : [a, b] \to \mathbb{R}$ is differentiable and $f'(a) < L < f'(b)$ or $f'(b) < L < f'(a)$.
Then, there exists some c with $f'(c) = L$. (we do not assume that f is continuous)

Rolle's Theorem:
Assume $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b) and $f(a) = f(b)$.
Then, there exists some $c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem:
Assume $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable on (a, b).
Then, there exists some $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

6 Sequences of functions

6.1 Pointwise convergence

Consider $f_n : A \to \mathbb{R}$.
(f_n) **converges pointwise** to $f : A \to \mathbb{R}$ is for all fixed $x \in A \quad \lim f_n(x) = f(x)$
Thus, for each fixed $x \in A$ we have

$$\forall \epsilon > 0 \ \ \exists N_{\epsilon,x} \in \mathbb{N} \ \text{ s.t. } \ n \geq N_{\epsilon,x} \implies |f_n(x) - f(x)| < \epsilon$$

6.2 Uniform convergence

(f_n) **converges uniformly** to $f : A \to \mathbb{R}$ if

$$\forall \epsilon > 0 \ \ \exists N_{\epsilon} \in \mathbb{N} \ \text{ s.t. } \ n \geq N_{\epsilon} \implies |f_n(x) - f(x)| < \epsilon \ \ \forall x \in A$$

Uniform means that N_ϵ is independent of $x \in A$.
$f_n \to f$ uniformly means that for $n \geq N$ the graph of f_n is within an **ϵ-ribbon** around f.

If $f_n \to f$ uniformly on A, then $\lim \left(\sup_{x \in A} |f_n(x) - f(x)| \right) = 0$

If for all n sufficiently large $|f_n - f|$ is bounded on A and $\lim \left(\sup_{x \in A} |f_n(x) - f(x)| \right) = 0$,
then $f_n \to f$ uniformly on A.

Let $f_n : A \to \mathbb{R}$. If $f_n \to f$ uniformly on A and f_n is continuous at c, then f is continuous at c.

Cauchy criterion for functions: f_n converges uniformly on A if and only if

$$\forall \epsilon > 0 \quad \exists N_\epsilon \in \mathbb{N} \quad \text{s.t.} \quad n, m \geq N \implies |f_n(x) - f_m(x)| < \epsilon \quad \forall x \in A$$

If $f_n \to f$ uniformly on A, then $\quad f_n$ continuous on A for all $n \implies f$ continuous on A

Assume that

1. $f_n : [a, b] \to \mathbb{R}$ differentiable for all n
2. $f_n' \to g$ uniformly on $[a, b]$
3. $f_n(x_0)$ converges for some $x_0 \in [a, b]$

Then there exists a differentiable $f : [a, b] \to \mathbb{R}$ such that $f_n \to f$ uniformly and $f' = g$.
(Under these conditions, limit and derivative can be swapped)

7 Series of functions

$\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on A if and only if

$$\forall \epsilon > 0 \quad \exists N_\epsilon \in \mathbb{N} \quad \text{s.t.} \quad n, m \geq N \implies |f_{m+1}(x) + \ldots + f_n(x)| < \epsilon \quad \forall x \in A$$

Weierstrass Test:
Assume that $|f_n(x)| < C_n$ for all $x \in A$ and $\sum\limits_{n=1}^{\infty} C_n$ converges. Then $\sum\limits_{n=1}^{\infty} f_n$ converges on A.

Assume that $\sum\limits_{n=1}^{\infty} f_n \to f$ uniformly on A and f_n is continuous on A for all n.
Then f is continuous on A.

Assume:

1. $f_n : [a, b] \to \mathbb{R}$ is differentiable for all n
2. $\sum\limits_{n=1}^{\infty} f_n' \to g$ uniformly on $[a, b]$
3. $\sum\limits_{n=1}^{\infty} f_n(x_0)$ converges for some $x_0 \in [a, b]$

Then there exists a differentiable $f : [a, b] \to \mathbb{R}$ s.t. $\sum\limits_{n=1}^{\infty} f_n \to f$ uniformly and $f' = \sum\limits_{n=1}^{\infty} f_n'$

7.1 Power series

General form of a **power series**: $\sum\limits_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots$

$$\sum_{n=0}^{\infty} a_n c^n \text{ converges at } c \neq 0 \implies \sum_{n=0}^{\infty} |a_n x^n| \text{ converges for } |x| < |c|$$

There exists $R \geq 0$ (the **radius of convergence**) such that

$$|x| < R \implies \text{PS converges at } x \qquad |x| > R \implies \text{PS diverges at } x$$

Root test: if $L = \lim \sqrt[n]{|a_n|}$ exists, then $R = \dfrac{1}{L}$

Ratio test: if $L = \lim \left|\dfrac{a_{n+1}}{a_n}\right|$ exists, then $R = \dfrac{1}{L}$

$$\sum_{n=0}^{\infty} |a_n c^n| \text{ convergent} \implies \sum_{n=0}^{\infty} a_n x^n \text{ uniformly convergent on } [-|c|, |c|]$$

$\sum\limits_{n=0}^{\infty} a_n x^n$ is a continuous function on $(-R, R)$.

$$\sum_{n=0}^{\infty} |a_n R^n| \text{ convergent} \implies \sum_{n=0}^{\infty} a_n x^n \text{ uniformly convergent and continuous on } [-R, R]$$

Summation by parts: If $s_n = u_1 + u_2 + \ldots + u_n$, then $\sum\limits_{k=1}^{n} u_k v_k = s_n v_{n+1} + \sum\limits_{k=1}^{n} s_k (v_k - v_{k+1})$

Abel's Lemma:

Assume that (u_n) and (v_n) satisfy $|u_1 + \ldots + u_n| \leq C$ and $0 \leq v_{n+1} \leq v_n$ for all $n \in \mathbb{N}$.

Then, $\left|\sum\limits_{k=1}^{n} u_k v_k\right| \leq C v_1 \quad \forall n \in \mathbb{N}$

Abel's Theorem:

1. PS converges at $x = R \implies$ PS converges uniformly on $[0, R]$
2. PS converges at $x = -R \implies$ PS converges uniformly on $[-R, 0]$

$$\sum_{n=0}^{\infty} a_n x^n \text{ convergent on } (-R, R) \implies \sum_{n=0}^{\infty} n a_n x^{n-1} \text{ convergent on } (-R, R)$$

For any power series with radius R we have $\left(\sum\limits_{n=0}^{\infty} a_n x^n\right)' = \sum\limits_{n=0}^{\infty} n a_n x^{n-1} \qquad \forall x \in (-R, R)$

7.2 Taylor series

The **Taylor series** of f around $x = 0$ is given by $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Partial sum $s_n(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$, remainder $E_n(x) = f(x) - s_n(x)$

For $n \in \mathbb{N}$ and $x > 0$, $\exists c \in (0, x)$ such that $E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ where c depends on n and x.

If $x < 0$, then $c \in (x, 0)$

Assume that

- $x > 0$ and $h(t)$ is $n+1$ times differentiable on $[0, x]$
- $h(x) = 0$ and $h^{(k)}(0) = 0$ for all $k = 0, \ldots, n$

Then $h^{(n+1)}(c) = 0$ for some $c \in (0, x)$

The Taylor series of f around $x = a$ is given by $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$

with remainder $E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}$

8 Integrals

8.1 Definition

A **partition** of $[a, b]$ is a set of the form $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$

Let f be bounded and P be a partition of $[a, b]$.

The **lower sum** $L(f, P)$ of f with respect to p is $\sum\limits_{k=1}^{n} m_k (x_k - x_{k-1})$

where $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$

The **upper sum** $U(f,P)$ of f with respect to p is $\sum_{k=1}^{n} M_k(x_k - x_{k-1})$

where $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$

For any partition P, we have $L(f,P) \leq U(f,P)$

Q is called a **refinement** of P if $P \subseteq Q$, provided that they are partitions of the same interval.

If $P \subseteq Q$, then

$$L(f,P) \leq L(f,Q) \qquad U(f,P) \geq U(f,Q) \qquad U(f,Q) - L(f,Q) \leq U(f,P) - L(f,P)$$

For any two partitions P_1 and P_2, we have $L(f,P_1) \leq U(f,P_2)$

Assume $[a,b]$ is bounded. Let $\mathcal{P}$ denote the collection of all partitions of $[a,b]$. Then

$$U(f) = \inf\{U(f,P) : P \in \mathcal{P}\} \qquad L(f) = \sup\{L(f,P) : P \in \mathcal{P}\} \qquad L(f) \leq U(f)$$

A bounded function $f : [a,b] \to \mathbb{R}$ is called **Riemann integrable** if $U(f) = L(f)$.

Notation: $\int_a^b f = U(f) = L(f)$ or $\int_a^b f(x)dx = U(f) = L(f)$

f is integrable $\iff$ for all $\epsilon > 0$ there exists a partition P_ϵ such that $U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon$

f continuous on $[a,b] \implies f$ integrable on $[a,b]$

Monotone functions are integrable.

8.2 Properties of integrals

Lemma: $x \leq y + \epsilon \ \ \forall \epsilon > 0 \implies x \leq y$ (the inequality on the left can also be strict)

Split property: Let $f : [a,b]$ be bounded and $c \in (a,b)$. Then

1. f integrable on $[a,b] \iff f$ integrable on $[a,c]$ and $[c,b]$
2. $\int_a^b f = \int_a^c f + \int_c^b f$

If f, g are integrable on $[a,b]$, then

1. $\int_a^b f = -\int_b^a f$ and $\int_c^c f = 0 \ \ \forall c \in [a,b]$ (by definition)
2. $f+g$ integrable and $\int_b^a (f+g) = \int_b^a f + \int_b^a g$
3. kf integrable and $\int_b^a kf = k\int_b^a f \ \ \forall k \in \mathbb{R}$
4. $m \leq f(x) \leq M \implies m(b-a) \leq \int_a^b f \leq M(b-a)$
5. $f(x) \leq g(x) \ \ \forall x \in [a,b] \implies \int_a^b f \leq \int_a^b g$
6. $|f|$ is integrable and $\left|\int_a^b f\right| \leq \int_a^b |f|$

8.3 Fundamental Theorem of Calculus

FTC Part 1: If f is integrable on $[a,b]$, F is differentiable on $[a,b]$ and $F'(x) = f(x) \ \ \forall x \in [a,b]$, then $\int_a^b f = F(b) - F(a)$

FTC Part 2: Let f be integrable on $[a,b]$ and define $F(x) = \int_a^x f(t)dt$ where $x \in [a,b]$. Then

1. F is uniformly continuous on $[a, b]$
2. If f is continuous at c, then F is differentiable at c and $F'(c) = f(c)$

9 Exam Prep

NOTE: POSSIBLY OUTDATED

9.1 Problems

6 problems (chapter $1, 2, 3, (4, 5), 6, 7$) 15 points each + 10 free points

1. Supremum/infimum
2. (Monotone) Convergence
3. Compact sets (hard)
4. Continuous/differentiable, Mean (or Intermediate) Value Theorem
5. Sequences of functions, uniform convergence (hard)
6. Integrable, constructing partitions, FTC (easy, but done poorly by most students)

9.2 Tips

- Name + student number on each page
- Start by reading all questions.
- Roughly 15 minutes per question.
- Consise answers (examples on Brightspace)
- Follow the hints when given.
- Doing 4 problems well is better than 6 problems poorly.
- Do not state a lemma when asked for the definition.
- Always state assumption and conclusions when formulating a theorem.
- You can refer to a theorem by its number (don't), its name, or by formulating its contents.
- When applying a theorem, make sure that the conditions are satisfied.
- Standard limits are common knowledge and don't need a proof
- Polynomials, e^x, $\ln(x)$, trig functions and $\arctan(x)$ are continuous/differentiable
- Use theorems instead of definitions
- For piecewise functions, you can use different theorems for different pieces.
- Never forget the (reverse) triangle inequality
- Induction is not always needed
- Neatness does not matter as much
- For the defintion of compactness, the sequence is arbitrary, and elements can repeat